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Chromatic number of prime distance graphs

M. Voigt*, H. Walther

Technische Hochschule, Institut für Mathematik, 98684 Germany

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Abstract

For any set D of positive integers, the distance graph $G(D) = G(V, E)$ is the graph with vertex set $V(G) = \mathbb{Z}$ and edge set $E(G) = \{(u, v): |u - v| \in D\}$. In Research Problem 77 (Discrete Math. 69 (1988) 105–106) Eggleton, Erdös and Skilton propose the problem to determine all minimal subsets D of the prime numbers such that graph $G(D)$ is 4-chromatic. In the present paper this problem is solved for 4-element prime sets D.

1. Introduction

Let $D = \{d_1, d_2, d_3, \ldots, d_r\} \subseteq \mathbb{N}$ be a finite nonempty subset of the set $\mathbb{N}$ of all positive integers.

The graph $G(D) = G(V, E)$ with vertex set $V(G) := \mathbb{Z}$ ($\mathbb{Z}$ is the set of all integers) and edge set $E(G) := \{(u, v): |u - v| \in D\}$ is called the *distance graph* of the *distance set* D. $\chi(D)$ denotes the chromatic number of $G(D)$.

It is known [1, 5, 8] that:

- $\chi(D) = 4$ if D is the set $\mathbf{P}$ of all primes,
- $\chi(D) = 2$ if $D \subseteq \mathbf{P} \setminus \{2\}$,
- $\chi(D) \leqslant 3$ if $D \subseteq \mathbf{P} \setminus \{3\}$,
- $\chi(D) \in \{3, 4\}$ if $\{2, 3\} \subseteq D \subseteq \mathbf{P}$.

Therefore we investigate sets $D \subseteq \mathbf{P}$ with $\{2, 3\} \subseteq D$ only.

Eggleton et al. [3] gave the following conjecture: Let D be a subset of the set $\mathbf{P}$ ($\{2, 3\} \subseteq D$) of primes. Then $\chi(D) = 4$ if and only if D contains a twin of primes.

It is easy to see that $\chi(D) = 4$ if D contains a twin of primes. The second part of this conjecture is disproved in [2] and in [8] by counterexamples.

* Corresponding author.

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In 1988, an update on the above conjecture was published (see [2]). Eggleton et al. now formulated the problem in the following way: characterize all sets of primes with $\chi(D) = 3$.

We prove the following theorem: *There are exactly eight pairs p, q of primes* ($p \geqslant 7$, $q > p + 2$) *with* $\chi(2, 3, p, q) = 4$.

2. Definitions

We investigate colourings of $\mathbb{Z}$ with three colours. These three colours are denoted by a, b and c.

Definition 2.1. A colouring $f: \mathbb{Z} \to \{\mathrm{a, b, c}\}$ is called *d_i-consistent*, if $f(v) \neq f(v + d_i)$ holds for all $v \in \mathbb{Z}$. A colouring f is *D-consistent* if f is d_i-consistent for each $i \in \{1, 2, \ldots, r\}$.

Definition 2.2. A colouring $f: \mathbb{Z} \to \{\mathrm{a, b, c}\}$ is called *periodically with period* λ if $f(v) = f(v + \lambda)$ for all $v \in \mathbb{Z}$. Such a colouring is denoted by P_λ.

In what follows we obtain certain "sections of colours" which can occur in (2, 3)-consistent 3-colourings. We choose without loss of generality (w.l.o.g.) always the colour a as the initial colour of such finite *colour-sections* or briefly *sections* for our description.

Let $\mathbb{Z}^{\mathrm{a}} := \{v \mid v \in \mathbb{Z}$ with $f(v) = \mathrm{a}$ and $f(v - 1) \neq \mathrm{a}\}$. In Fig. 1 we have: $\{-14, -9, -5, 0, 5, 10, 14\} \subseteq \mathbb{Z}^{\mathrm{a}}$.

Definition 2.3. A sequence $S_l := (f^1, \ldots, f^l)$ of colours f^i with $l > 1$, $f^i \in \{\mathrm{a, b, c}\}$, $i \in \{1, \ldots, l\}$, $f^1 = \mathrm{a}$, $f^l \neq \mathrm{a}$, is called a *(colour-)section* of length l.

A colour-section S_l ($l \geqslant 3$) with $f^i \neq \mathrm{a}$ $\forall i = 3, \ldots, l$ is called an *elementary (colour-)section* of length l.

For instance, (a, a, b, c, c, a, b, b, c) is a section of length 9, but this section is not elementary, whereas (a, a, b, c, c) is an elementary section of length 5.

A colouring $f: \mathbb{Z} \to \{\mathrm{a, b, c}\}$ *contains the section* $S_l = (f^1, \ldots, f^l)$ if there is a vertex $v \in \mathbb{Z}^{\mathrm{a}}$ with $f(v) = f^1 = \mathrm{a}$, $f(v + 1) = f^2, \ldots, f(v + l - 1) = f^l \neq \mathrm{a}$, $f(v + l) = \mathrm{a}$. We denote this also by $S_l \in f$ (S_l is a section of f). In Fig. 1 we have for instance: (a, a, b, c, c) $\in f$, (a, a, b, c, c, a, b, b, c) $\in f$.

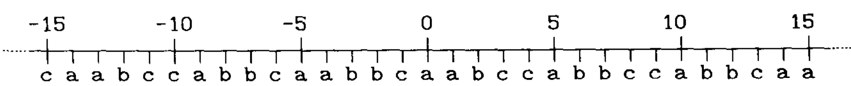

Fig. 1.

3. (2, 3)-consistency of 3-colourings

Elementary sections are investigated which may occur in a (2, 3)-consistent 3-colouring f.

We fix w.l.o.g. the ordering of colours for any elementary section: a, b, c. We get the same ordering of colours for all elementary sections of f and we obtain that there exist exactly five elementary sections, namely:

$$A_4 := (\mathrm{a, b, b, c}), \qquad A_5 := (\mathrm{a, b, b, c, c}), \qquad B_5 := (\mathrm{a, a, b, c, c}),$$

$$C_5 := (\mathrm{a, a, b, b, c}) \quad \text{and} \quad A_6 := (\mathrm{a, a, b, b, c, c}).$$

Theorem 3.1 (Voigt [6]). *A 3-colouring f is (2, 3)-consistent iff the elementary sections of f are none other than A_4, A_5, B_5, C_5 or A_6 and no section S_l of f contains: A_4A_4, C_5A_4, C_5A_5 or A_4A_5.*

Corollary 3.2. *Let f be a (2, 3)-consistent 3-colouring of $\mathbb{Z}$. Then there exists at least one v^0 with $v^0 \in \mathbb{Z}^{\mathrm{a}}$ in an arbitrary interval of length six.*

4. p-consistency of (2, 3)-consistent 3-colourings

Specifications. Let S_l be a section of a 3-colouring f of $\mathbb{Z}$. The first elementary section of S_l is denoted by E_F, the last by E_L, the last before S_l by E_{F-1} and the next after S_l by E_{L+1} (see Fig. 2).

Theorem 4.1. *A (2, 3)-consistent 3-colouring is n-consistent ($n \in \mathbb{N}$, $n \geqslant 4$) iff the following five conditions are fulfilled:*

(i) $\forall S_l \in f \Rightarrow l \neq n$,

(ii) $\forall S_{n+2} \in f \Rightarrow E_F \neq C_5$ *or* $E_{L+1} \neq A_5$,

(iii) $\forall S_{n-2} \in f \Rightarrow E_F \neq A_5$ *or* $E_{L+1} \neq C_5$,

(iv) $\forall S_{n-1} \in f \Rightarrow E_{F-1} \in \{A_4, C_5\}$ *and* $E_{L+1} \in \{A_4, A_5\}$,

(v) $\forall S_{n+1} \in f \Rightarrow E_F \in \{A_4, A_5\}$ *and* $E_L \in \{A_4, C_5\}$.

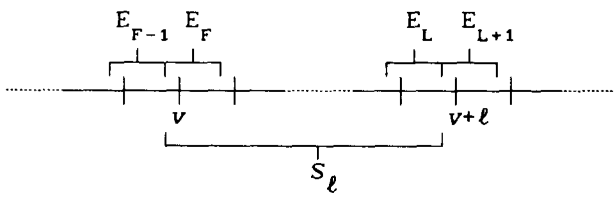

Fig. 2.

Main idea of the proof. (1) (i), ..., (v) follow from n-consistency. Assume the contrary for any of the five conditions. The contradiction follows immediately.

(2) n-consistency follows from (i), ..., (v). We have to show: $\forall v_1, v_2 \in \mathbb{Z}$ with $v_2 - v_1 = n$ it holds: $f(v_1) \neq f(v_2)$. We look for integers $v_1^0, v_2^0 \in \mathbb{Z}^a$ "near" v_1 and v_2, respectively. Such an integer $v_i^0 \in \mathbb{Z}^a$ exists in each interval of length 6 (see Corollary 3.2). Therefore we have:

$$\exists v_1^0 \in \mathbb{Z}^a,\ x_1 \in \{-3, \dots, +2\} \text{ with } v_1 = v_1^0 + x_1.$$

It is favourable for the proof to choose the interval for x_2 as a function of the value of x_1.

$$\exists v_2^0 \in \mathbb{Z}^a,\ x_2 \in \{x_{2,1}, \dots, x_{2,2}\} \text{ with } x_{2,2} - x_{2,1} = 5 \text{ and } v_2 = v_2^0 + x_2.$$

It follows that $v_2^0 - v_1^0 = v_2 - v_1 + x_1 - x_2 = n + x_1 - x_2$ and $v_2^0 = v_1^0 + n + x_1 - x_2$. From (i) we immediately obtain $x_2 \neq x_1$.

- Let $x_1 = -3$. Put $x_2 \in \{-5, -4, -2, -1, 0\}$.
- Let $x_1 = -2$. Put $x_2 \in \{-3, -1, 0, 1, 2\}$.
- Let $x_1 = -1$. Put $x_2 \in \{-3, -2, 0, 1, 2\}$.
- Let $x_1 = 0$. Put $x_2 \in \{-3, -2, -1, 1, 2\}$.
- Let $x_1 = 1$. Put $x_2 \in \{-1, 0, 2, 3, 4\}$.
- Let $x_1 = 2$. Put $x_2 \in \{-1, 0, 1, 3, 4\}$.

We have to consider $f(v_1)$ and $f(v_2)$ for all 30 cases. Taking the conditions (i), ..., (v) into consideration we always obtain the inequality $f(v_1) \neq f(v_2)$.

In what follows, we discuss the cases for $x_1 = -2$ to illustrate the argumentation. Hence, suppose $x_1 = -2$. It follows that

$$f(v_1) = \begin{cases} \mathrm{b} & \text{if } A_4 \text{ or } C_5 \text{ ends in } v_1^0 - 1, \\ \mathrm{c} & \text{otherwise.} \end{cases}$$

Case 1: $x_2 = -3, f(v_2) = \mathrm{b}$. It follows that $v_2^0 = v_1^0 + n + 1$ and A_4 or A_5 begin in v_1^0 (see (v)). Therefore we have $f(v_1^0 + 1) = \mathrm{b}$ and $f(v_1^0 - 2) = f(v_1) \neq \mathrm{b}$ (3-consistency).

Case 2: $x_2 = -1, f(v_2) = \mathrm{c}$. It follows that $v_2^0 = v_1^0 + n - 1$ and A_4 or C_5 end in $v_1^0 - 1$ (see (iv)). Thus we have $f(v_1) = \mathrm{b} \neq \mathrm{c}$.

Case 3: $x_2 = 0, f(v_2) = \mathrm{a} \neq f(v_1)$.

Case 4: $x_2 = 1$,

$$f(v_2) = \begin{cases} \mathrm{b} & \text{if } A_4 \text{ or } A_5 \text{ begins in } v_2^0, \\ \mathrm{a} & \text{otherwise.} \end{cases}$$

We show: $f(v_1)$ and $f(v_2)$ cannot be equal to "b" simultaneously. Let $f(v_1) = \mathrm{b}$.

(1) A_4 ends in $v_1^0 - 1$. We have $v_1^1 := v_1^0 - 4 \in \mathbb{Z}^a$ and $v_2^0 - v_1^1 = n + 1$. It follows from (v) that A_4 or C_5 ends in $v_2^0 - 1$ and consequently $f(v_2^0 - 2) = \mathrm{b}$. Because of 3-consistency we have $f(v_2^0 + 1) = f(v_2) \neq \mathrm{b}$.

(2) C_5 ends in $v_1^0 - 1$. We have $v_1^1 := v_1^0 - 5 \in \mathbb{Z}^a$ and $v_2^0 - v_1^1 = n + 2$. It follows from (ii) that A_5 does not begin with v_2^0. Let us assume A_4 begins in v_2^0. We have $v_2^1 := v_2^0 + 4 \in \mathbb{Z}^a$ and $v_2^1 - v_1^0 = n + 1$. It follows from (v) that $f(v_1^0 + 1) = \mathrm{b}$ and $f(v_1^0 - 2) = f(v_1) \neq \mathrm{b}$ because of 3-consistency. This contradiction implies: A_4 does not begin in v_2^0. Therefore we have $f(v_2) \neq \mathrm{b}$.

Case 5: $x_2 = 2, f(v_2) = \mathrm{b}$. Now $v_2^0 - v_1^0 = n - 4$. We have to show: A_4 or C_5 does not end in $v_1^0 - 1$. First assume A_4 ends in $v_1^0 - 1$. It follows that $v_1^1 := v_1^0 - 4 \in \mathbb{Z}^a$ and $v_2^0 - v_1^1 = n$—contradicting (i). Then assume C_5 ends in $v_1^0 - 1$. It now follows that $v_1^1 := v_1^0 - 5 \in \mathbb{Z}^a$ and $v_2^0 - v_1^1 = n + 1$. Thus A_4 or A_5 begins with v_1^1 (see (v)) — contradiction to assumption.

The remaining 25 cases can be handled in an analogous way. □

Corollary 4.2. *A 3-colouring f, consisting of elementary sections A_6 and B_5 only, is (2, 3, n)-consistent if it contains no section of the length $n - 1$, n and $n + 1$.*

Theorem 4.1 and Corollary 4.2 permit to investigate the n-consistency of 3-colourings.

5. Important results

At first, some known results are presented.

Lemma 5.1 (Walther [8]). *Let $p, q \in \mathbf{P}$, $p \geqslant 7$, $q = p + 8$. Then $\chi(2, 3, p, q) = 4$ holds if and only if $p \in \{11, 23, 29\}$.*

Lemma 5.2 (Voigt [6]). *Let $D = \{2, 3, p, q\}$. $\chi(D)$ is equal to 4 in case of the following five pairs of primes $\{p, q\}$:*

(1) $p = 11, q = 23$;
(2) $p = 11, q = 37$;
(3) $p = 11, q = 41$;
(4) $p = 17, q = 29$;
(5) $p = 23, q = 41$.

Thus, we have eight sets of four primes $D = \{2, 3, p, q\}$ $(p \geqslant 7, q > p + 2)$ with $\chi(D) = 4$ and we can show that additional sets with this property do not exist.

Lemma 5.3 (Voigt and Walther [7]). *Let $\Delta \in \mathbb{N}$, $\Delta \geqslant 10$. We have $\chi(2, 3, u, u + \Delta) = 3$ for all $u \in \mathbb{N}$ and $u \geqslant \Delta^2 - 6\Delta + 3$.*

Lemma 5.4 (Walther [8]). *Each d-consistent periodical colouring P_λ is also $(\lambda + d)$- and $(\lambda - d)$-consistent.*

Consequently, the colourings $P_{q-3}, P_{q-2}, P_{q+2}, P_{q+3}$ consisting of the sections B_5 and A_6 only, are (2, 3, q)-consistent.

Theorem 5.5. *Let $D = \{2, 3, p, q\}$ be a set of primes with $p \geqslant 7$ and $q > p + 2$. Then $\chi(D) = 4$ holds if and only if*

$$(p, q) \in \{(11, 19), (11, 23), (11, 37), (11, 41), (17, 29), (23, 31), (23, 41), (29, 37)\}.$$

The idea of the proof of this theorem is described in the following sections.

6. Construction of colourings

We construct 3-colourings consisting of the elementary sections B_5 and A_6 only. First we fix the period λ and the number α of sections B_5 in P_λ. Note that $\lambda - 5\alpha$ must be divisible by 6. Consequently we obtain the number β of sections A_6 by $\beta = (\lambda - 5\alpha)/6$.

We assign the sections A_6 to the sections B_5 "evenly"(see algorithm (#)).

Examples. $\lambda = 111$.

(1) $\alpha = 3$, $\beta = 16$: $P_{111} := B_5A_6A_6A_6A_6A_6B_5A_6A_6A_6A_6A_6B_5A_6A_6A_6A_6A_6A_6$

$$= B_5(A_6)^5B_5(A_6)^5B_5(A_6)^6.$$

(2) $\alpha = 9$, $\beta = 11$:
$P_{111} := B_5A_6B_5A_6B_5A_6B_5A_6B_5A_6A_6B_5A_6B_5A_6B_5A_6B_5A_6A_6$.

(3) $\alpha = 15$, $\beta = 6$:
$P_{111} := B_5B_5B_5A_6B_5B_5A_6B_5B_5B_5A_6B_5B_5A_6B_5B_5B_5A_6B_5B_5A_6$.

The largest integer less than or equal to x is denoted by $\lfloor x \rfloor$.

Algorithm (#) to assign evenly

(i) the numbers $z_1, \ldots, z_\alpha$ are defined by the following algorithm:

(0) α – number of sections B_5,

β – number of sections A_6,

$$\gamma = \frac{\beta}{\alpha} - \left\lfloor \frac{\beta}{\alpha} \right\rfloor (< 1),$$

$$z = \left\lfloor \frac{\beta}{\alpha} \right\rfloor,$$

z_i – number of sections A_6 assigned to the ith section B_5.

(1) $R_0^1 := 0, \qquad R_1^1 := \gamma,$

$R_1^2 := 0,$

$z_1 := z = \left\lfloor \dfrac{\beta}{\alpha} \right\rfloor,$

$i := 2.$

(2) $\bar{R}_i^1 := R_{i-1}^1 + \gamma, \qquad R_i^2 := \begin{cases} 0, & \text{if } \bar{R}_i^1 < 1, \\ 1, & \text{otherwise,} \end{cases}$

$z_i := z + R_i^2 = \left\lfloor \dfrac{\beta}{\alpha} \right\rfloor + R_i^2,$

$R_i^1 := \bar{R}_i^1 - R_i^2,$

$i := i + 1,$

for $i \leqslant \alpha$ go to (2), otherwise go to END.

END:

(ii) the periodical colouring P_λ is defined by

$$P_\lambda := B_5(A_6)^{z_1} B_5(A_6)^{z_2} \cdots B_5(A_6)^{z_{\alpha-1}} B_5(A_6)^{z_\alpha}.$$

Let r and j be arbitrary natural numbers. We consider a section S_L consisting of r successive sections B_5 and the sections A_6 assigned by (#) to this sections B_5. Let the section S_L begin with the $(j+1)$st section B_5. We obtain:

$$L = 5r + 6 \sum_{i=j+1}^{j+r} z_i.$$

$Z := \sum_{i=j+1}^{j+r} z_i$ is the number of sections A_6 from S_L. It is easy to see that

$$Z = rz + \sum_{i=j+1}^{j+r} R_i^2 = rz + \lfloor R_j^1 + r\gamma \rfloor.$$

It follows from $0 \leqslant R_j^1 < 1$ that

$$rz + \lfloor r\gamma \rfloor \leqslant Z \leqslant rz + \lfloor r\gamma \rfloor + 1. \qquad (*)$$

Let P_λ be a periodical 3-colouring constructed by the algorithm given above.

Definition 6.1. Let

$$L_{\min}^r := \min_{j \in \mathbb{N}} \left(5r + 6 \sum_{i=j+1}^{j+r} z_i \right), \qquad L_{\max}^r := \max_{j \in \mathbb{N}} \left(5r + 6 \sum_{i=j+1}^{j+r} z_i \right)$$

be the minimal (maximal) length of sections of P_λ consisting of r successive sections B_5 and their assigned sections A_6.

It is easy to see that $L^r_{\min} \leqslant L^r_{\max} \leqslant L^r_{\min} + 6$. We obtain from ($*$):

$$L^r_{\min} = 5r + 6[rz + \lfloor r\gamma \rfloor] = 6\left\lfloor r\frac{\beta}{\alpha} \right\rfloor + 5r \qquad (**)$$

and therefore

$$L^r_{\max} \leqslant 6\left\lfloor r\frac{\beta}{\alpha} \right\rfloor + 5r + 6.$$

Furthermore it follows from ($*$):

Lemma 6.2. *Let A_L be an arbitrary section of the colouring P_λ consisting of r successive sections B_5 and their assigned (by (#)) sections A_6. Then it holds $L^r_{\min} \leqslant L \leqslant L^r_{\max}$.*

7. p-consistency of a colouring $\mathbf{P}_\lambda$

The periodical colouring P_λ is constructed by using the algorithm (#).

Any section S_L of P_λ with a length $L \geqslant L^1_{\max}$ contains at least one section B_5. Any section S_L of P_λ with a length $L \leqslant L^3_{\min}$ contains at most three sections B_5. Therefore, we have $L \equiv 3, 4$ or 5 modulo 6 for any section S_L with $L^1_{\max} \leqslant L \leqslant L^3_{\min}$. Consequently, such a colouring is p-consistent for all $p \equiv 1 (\mathrm{mod}\, 6)$ and $L^1_{\max} < p < L^3_{\min}$ by Corollary 4.2. It follows that $L^1_{\max} + 2 \leqslant p \leqslant L^3_{\min} - 2$ because of $L^1_{\max} \equiv -1 (\mathrm{mod}\, 6)$, $p \equiv 1 (\mathrm{mod}\, 6)$ and $L^3_{\min} \equiv 3 (\mathrm{mod}\, 6)$. Furthermore, we obtain that such a colouring P_λ is p-consistent for all $p \equiv 1 (\mathrm{mod}\, 6)$ with

$$L^r_{\max} + 2 \leqslant p \leqslant L^{r+2}_{\min} - 2, \quad r = 6j + 1 \text{ and } j = 0, 1, 2, \ldots.$$

In the case $p \equiv -1 (\mathrm{mod}\, 6)$ we obtain in an analogous way the consistency for all p with

$$L^r_{\max} + 2 \leqslant p \leqslant L^{r+2}_{\min} - 2, \quad r = 6j + 3 \text{ and } j = 0, 1, 2, \ldots.$$

Now we derive from ($**$):

Lemma 7.1. *A periodical 3-colouring P_λ constructed by algorithm (#) is p-consistent*

(1) *for all $p \in \mathbf{P}$, $p \equiv 1 (\mathrm{mod}\, 6)$, $r = 6j + 1$ and*

$$6\left\lfloor (6j+1)\frac{\beta}{\alpha} \right\rfloor + 5(6j+1) + 8 \leqslant p \leqslant 6\left\lfloor (6j+3)\frac{\beta}{\alpha} \right\rfloor + 5(6j+3) - 2$$

with $j = 0, 1, 2, \ldots$;

(2) *for all $p \in \mathbf{P}$, $p \equiv -1 (\mathrm{mod}\, 6)$, $r = 6j + 3$ and*

$$6\left\lfloor (6j+3)\frac{\beta}{\alpha} \right\rfloor + 5(6j+3) + 8 \leqslant p \leqslant 6\left\lfloor (6j+5)\frac{\beta}{\alpha} \right\rfloor + 5(6j+5) - 2$$

with $j = 0, 1, 2, \ldots$.

8. Sequences of colourings

We fix an integer $\alpha = \alpha_0$ and a period λ such that $\lambda - 5\alpha$ is divisible by 6: $\beta_0 := (\lambda - 5\alpha)/6$. Let

$$\alpha_i = 6i + \alpha \quad \text{for } i = 0, 1, \ldots, i_{\max}$$

(for the choice of $i_{\max}$ see below) and

$$\beta_i := \frac{\lambda - 5\alpha_i}{6} = \frac{\lambda - 5(6i+\alpha)}{6} = \beta_{i-1} - 5 \quad (i = 1, \ldots, i_{\max}).$$

We investigate periodical colourings $P_\lambda^i = P_\lambda(\alpha_i, \beta_i)$, $i = 0, 1, \ldots, i_{\max}$, constructed by algorithm (#). We have to ensure that $\beta_{i_{\max}} \geqslant 0$.

It is easy to see (from (**)) that

$$L^r_{\max}(i) = 6\left\lfloor r\frac{\beta_i}{\alpha_i} \right\rfloor + 5r + 6,$$

$$L^{r+2}_{\min}(i) = 6\left\lfloor (r+2)\frac{\beta_i}{\alpha_i} \right\rfloor + 5r + 10.$$

For a given r we fix $i_{\max}$ to be the smallest i with $r\beta_i/\alpha_i < 1$.

The following lemma immediately results from Lemma 7.1.

Lemma 8.1. *A sequence P_λ^i of colourings contains p-consistent colourings*

(1) *for all* $p \in \mathbf{P}$, $p \equiv 1 (\text{mod } 6)$, $r = 6j + 1$ *and*

$$6\left\lfloor (6j+1)\frac{\beta_i}{\alpha_i} \right\rfloor + 5(6j+1) + 8 \leqslant p \leqslant 6\left\lfloor (6j+3)\frac{\beta_i}{\alpha_i} \right\rfloor + 5(6j+3) - 2$$

with $i = 0, 1, 2, \ldots, i_{\max}$, $j = 0, 1, 2, \ldots$;

(2) *for all* $p \in \mathbf{P}$, $p \equiv -1 (\text{mod } 6)$, $r = 6j + 3$ *and*

$$6\left\lfloor (6j+3)\frac{\beta_i}{\alpha_i} \right\rfloor + 5(6j+3) + 8 \leqslant p \leqslant 6\left\lfloor (6j+5)\frac{\beta_i}{\alpha_i} \right\rfloor + 5(6j+5) - 2$$

with $i = 0, 1, 2, \ldots, i_{\max}$, $j = 0, 1, 2, \ldots$.

9. Chromatic number

Lemma 9.1. *There exists only a finite number of sets $D = \{2, 3, p, q\}$, $p, q \in \mathbf{P}$, $p \geq 7$ and $q > p + 2$, with $\chi(D) = 4$.*

Idea of the proof. We construct $(2, 3, p, q)$-consistent periodical 3-colourings P^i_λ. The parameters of such colourings are listed in Table 1. We have to distinguish four cases: (i) $p \equiv 1, q \equiv 1$; (ii) $p \equiv 1, q \equiv -1$; (iii) $p \equiv -1, q \equiv -1$ and (iv) $p \equiv -1, q \equiv 1$ (see Table 1, columns 1 and 2). Each row of Table 1 represents a sequence of colourings (for $i = 0, \ldots, i_{\max}$ if α_i depends on (i)) or a single colouring with period λ given in column 4. The construction of these colourings is meaningful for primes q greater than or equal to the values in column 3.

The intervals of consistency for p are derived from Lemmas 7.1 and 8.1 (with fixed j, given in column 7), but the proofs are very sophisticated. We have carried out the case $p \equiv 1 (\mathrm{mod}\, 6)$, $q \equiv 1 (\mathrm{mod}\, 6)$ only. One can find the remaining three cases in [6].

Proof of Lemma 9.1. For $p \equiv 1 (\mathrm{mod}\, 6)$ and $q \equiv 1 (\mathrm{mod}\, 6)$, at first, we look at $(+)$ in Table 1.

(1) *We have to show $\beta_{i_{\max}} \geq 0$.*
For $i_{\max} \in \{0, 1\}$, we have $\beta_i = (q + 2 - 5(6i + 3))/6 \geq 0$, because of $q \geq 43$.
For $i_{\max} \geq 2$ we have $\beta_{i_{\max}} \geq 6i_{\max} - 8 \geq 0$ because of $i_{\max} \leq (q - 31)/66 + 1$.

(2) *Bounds for p.*

The sequence P^i_λ contains p-consistent colourings for all p with

$$6\left\lfloor \frac{\beta_i}{\alpha_i} \right\rfloor + 13 \leq p \leq 6\left\lfloor 3\frac{\beta_i}{\alpha_i} \right\rfloor + 13 \quad \text{by Lemma 8.1.}$$

– Lower bound for p: For $i = i_{\max}$ we get

$$p \geq \left\lfloor 6\frac{\beta_{i_{\max}}}{\alpha_{i_{\max}}} \right\rfloor + 13 = 13,$$

because of $\beta_{i_{\max}}/\alpha_{i_{\max}} < 1$.

– Upper bound for p: For $i = 0$ we obtain

$$p \leq 6\left\lfloor 3\frac{\beta_0}{\alpha_0} \right\rfloor + 13 = 6\left\lfloor \frac{q - 13}{6} \right\rfloor + 13 = q.$$

(3) *Overlapping of the "intervals of consistency".*

We have to show that the difference between the lower bound of the ith colouring and the upper bound of the $(i + 1)$th colouring is at most 6 (because $p \equiv 1 (\mathrm{mod}\, 6)$), this means

$$6\left\lfloor \frac{\beta_i}{6i + 3} \right\rfloor + 13 \leq 6\left\lfloor 3\frac{\beta_{i+1}}{6i + 9} \right\rfloor + 13 + 6 \quad \forall i = 0, 1, \ldots, i_{\max} - 1.$$

Table 1

$p \equiv$	$q \equiv$	$q \geqslant$	λ	α_i	$i_{\max}$	j	Interval of consistency of p	
1	1	43	$q+2$	$6i+3$	$\left\lfloor \frac{q-31}{66} \right\rfloor + 1$	0	$[13, q] \setminus \left\{ 6\left\lfloor \frac{q-13}{18} \right\rfloor + 7 \right\}$	(+)
			$q-2$	7	–	0	$\left\{ 6\left\lfloor \frac{q-13}{18} \right\rfloor + 7 \right\}$	(++)
1	-1	47	$q-2$	$6i+3$	$\left\lfloor \frac{q-35}{66} \right\rfloor + 1$	0	$[13, q-4] \setminus \left\{ 6\left\lfloor \frac{q-17}{18} \right\rfloor + 7 \right\}$	
			$q+2$	5	–	0	$\left\{ 6\left\lfloor \frac{q-17}{18} \right\rfloor + 7 \right\}$	
-1	-1	83	$q+2$	$6i+5$	$\left\lfloor \frac{q-33}{42} \right\rfloor + 1$	0	$[23, q] \setminus \left[6\left\lfloor 5\frac{q-53}{66} \right\rfloor + 29, 6\left\lfloor 3\frac{q-23}{30} \right\rfloor + 17 \right]$ [a]	
			$q-2$	9	–	0	$\left[6\left\lfloor 5\frac{q-53}{66} \right\rfloor + 29, 6\left\lfloor 5\frac{q-47}{54} \right\rfloor + 23 \right]$	
			$q+2$	17	–	1	$\left[6\left\lfloor 5\frac{q-47}{54} \right\rfloor + 29, 6\left\lfloor 3\frac{q-23}{30} \right\rfloor + 17 \right]$ [b]	
-1	1	97	$q-2$	$6i+7$	$\left\lfloor \frac{q-51}{42} \right\rfloor + 1$	0	$\left[23, 6\left\lfloor 5\frac{q-37}{42} \right\rfloor + 23 \right] \setminus \left[6\left\lfloor 5\frac{q-67}{78} \right\rfloor + 29, 6\left\lfloor 3\frac{q-37}{42} \right\rfloor + 17 \right]$	
			$q+2$	9	–	0	$\left[6\left\lfloor 5\frac{q-67}{78} \right\rfloor + 29, 6\left\lfloor 3\frac{q-37}{42} \right\rfloor + 17 \right]$	
			$q-2$	$6i+7$	$\left\lfloor \frac{q-58}{48} \right\rfloor + 1$	i	$\left[6\left\lfloor 5\frac{q-37}{42} \right\rfloor + 23, q-20 \right]$ [c]	

[a] For $q = 101$ the prime $p = 29$ is not contained in the corresponding interval of consistency.
[b] The primes $p = 47$ (for $q = 83$ and $q = 89$), $p = 59$ (for $q = 107$), $p = 71$ (for $q = 131$) and $p = 107$ (for $q = 197$) are not contained in the corresponding interval of consistency.
[c] For $q = 103$ the prime $p = 71$ is not contained in the corresponding interval of consistency.

Let $i \neq 0$, this means $i \in \{1, 2, \ldots, i_{\max} - 1\}$. It is sufficient to show

$$\frac{\beta_i}{6i+3} \leqslant \frac{\beta_{i+1}}{2i+3} + 1 = \frac{\beta_i - 5}{2i+3} + 1.$$

Table 2

p	q	$(2, 3, p, q)$-consistent 3-colourings
7	19	$P_{21} := C_5 A_6 A_5 B_5$
29	53	$P_{87} := (A_9)^3 B_5 (A_9)^2 A_4 C_5 B_5 (A_9)^2 A_5$
47	89	$P_{38} := A_5 C_5 A_6 B_5 A_6 B_5 A_6$

This is equivalent to

$$\beta_i \geqslant -3i + \frac{3}{2} + \frac{3}{2i}.$$

This inequality is correct for $i \geqslant 1$ because $\beta_i \geqslant 0$.

Let $i = 0$. The difference between the lower bound for $i = 0$ and the upper bound for $i = 1$ is for some q greater than 6, namely for all q with $\lfloor (q-13)/18 \rfloor = \lfloor (q-7)/18 \rfloor$. In these cases the integer $p = 6\lfloor (q-13)/18 \rfloor + 7$ is not contained in the intervals of consistency. Therefore, we have to find a p-consistent colouring for $p = 6\lfloor (q-13)/18 \rfloor + 7$. We look at $(++)$ in Table 1 and we obtain by Lemma 7.1 that the colouring is p-consistent for all p with $6\lfloor (q-37)/42 \rfloor + 13 \leqslant p \leqslant 6\lfloor (q-37)/14 \rfloor + 13$. It is easy to see that for $q \in \mathbf{P}$, $q \geqslant 43$ and $q \equiv 1 \pmod 6$ the integer $p = 6\lfloor (q-13)/18 \rfloor + 7$ is contained in this interval of consistency. □

Thus, there is only a finite number of pairs of primes p, q not contained in Table 1 (see also Lemma 5.3). We can find $(2, 3, p, q)$-consistent 3-colourings for all of them except the pairs from Lemmas 5.1 and 5.2. We use only the sections B_5 and A_6 or A_4 and B_5 for most of such colourings (see [6]). We do not give all these colourings here because most of them are easy to recognize. On the other hand it is not trivial to find the colourings for the three pairs p, q listed in Table 2.

Thus, the proof of Theorem 5.5 is complete. We have shown that there are exactly eight pairs of primes p, q $(p \geqslant 7, q > p + 2)$ with $\chi(2, 3, p, q) = 4$.

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